On Kelvin's ship-wave pattern

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When a concentrated pressure travels with constant velocity over the free surface of water, it carries with it a familiar pattern of ship waves. Let viscosity and surface tension be neglected, let the free-surface condition be linearized, let the depth of water be assumed infinite, and let initial transient effects be ignored. Then, as is well known, the wave motion everywhere can be found by standard methods in the form of a double integral. The wave pattern at a great distance behind the disturbance can be found by an application of the ordinary method of stationary phase, which shows that the wave amplitude is considerable inside an angle bounded by the two horizontal rays $\theta = \pm\theta_c$ from the disturbance, where $\theta_c = \sin^{-1}\frac{1}{3} \doteqdot 19\frac{1}{2}°$. But the method fails in two regions, near the track $\theta = 0$ of the pressure point, and near the critical lines $\theta = \pm\theta_c$.

These two regions are treated in the present paper. It is shown that near $\theta = 0$ the linearized surface elevation oscillates with indefinitely increasing amplitude and indefinitely decreasing wavelength. (This result holds only when the pressure is concentrated at a point and applied at the free surface.) Near the critical lines the surface elevation at a greater distance behind the pressure point can be expressed in terms of Airy functions, and this expression goes over into the known wave pattern inside the critical angle. It is shown that near the critical lines the crest length increases as the cube root of the distance, and that the separation between crests remains constant. Contour maps of the wave surface are given for three distances behind the moving pressure point.

1. Introduction

When a disturbance (e.g. a ship) travels on a water surface, it carries with it a familiar pattern of bow and stern waves which was first explained mathematically by Lord Kelvin (Sir W. Thomson 1891). Instead of ship waves he considered the waves generated by a prescribed pressure distribution moving with a constant velocity U and acting on the water surface. The magnitude of the pressure and the resulting wave slope were assumed to be so small that the equations of motion could be linearized; viscosity and surface tension were neglected. Under these assumptions it is sufficient (in principle) to calculate the waves due to a moving concentrated pressure point; the effect of a distribution of surface pressure can thence be found by integration. Kelvin showed, apparently by using his principle of stationary phase, that the characteristic wave pattern of a pressure point is the superposition of two sets of waves effectively confined between the two vertical planes $\theta = \pm\theta_c \doteqdot \pm 19\frac{1}{2}$, where the co-ordinates are taken as in § 2 below;

for a figure see Lamb (1932, §256). A simple argument using such ideas was given by Havelock (1908, p. 417), and by Lamb (1932, §256). All these results are valid at a distance of many wavelengths behind the pressure point; near the pressure point the principle of stationary phase is not applicable. It was found that near the critical lines $\theta = \pm\theta_c$ the amplitude is much larger than elsewhere at the same distance from the pressure point, and the simplest form of the principle of the stationary phase could not be used there. (Kelvin predicted infinite amplitudes on $\theta = \theta_c$; the amplitude variation on $\theta = \theta_c$ was first given correctly by Havelock (1908, see equation (4.1) below).) The study of the wave pattern near, but not actually on, a critical line is more difficult and was undertaken by Hogner (1923) who was able to write the surface elevation as the sum of a double and of a single integral, and to show that only the latter is significant at a large distance behind the pressure point. This is convenient because the application of the methods of steepest descents and of stationary phase to single integrals is simple. There are, however, some difficulties in his work. First, the single integral has one finite limit of integration, and this leads to complications (see p. 19 of his paper). Secondly, near the lines $\theta = \pm\theta_c$, where a single asymptotic expression in terms of Airy functions would be expected (cf. equation (4.12) below), Hogner's expression changes its form as θ passes through θ_c (Hogner, 1923, equations (82), (83)).

A new treatment of the problem has been given by Peters (1949). The surface elevation was obtained by him in a form similar to Hogner's, as the sum of a double and a single integral where again only the latter contributes significantly to the waves far behind the pressure point; but Peters's integral, unlike Hogner's, has both limits of integration at infinity. By an immediate application of the method of steepest descents he found expressions which are valid when θ is not near θ_c. An expression was also given for θ on and near $\theta = \theta_c$; but this is valid only when $N^{\frac{2}{3}}|\theta-\theta_c|$ is small, see §4 below.

While the work of Peters has thus reduced the problem of finding the waves far behind the pressure point to the asymptotic evaluation of a single integral containing a large parameter, there are two regions which have not yet received adequate treatment, and these will be considered in the present paper. One is the vicinity of the track of the pressure point where θ and z are small. The limits are non-uniform: we shall see by applying the method of steepest descents that the amplitude tends to ∞ as θ tends to 0 on $z = 0$ (but on $\theta = 0$ it is finite as z tends to 0). The other region is the neighbourhood of $\theta = \theta_c$, where an expression in terms of Airy functions will be given; here the method of Chester, Friedman & Ursell (1957) is applicable, and the expression is valid in some finite angle including the line $\theta = \theta_c$.

2. The elevation due to a travelling pressure point

Peters has calculated the velocity potential and the surface elevation $\zeta(x, y; U, g)$ above the mean water level. In this calculation the exact free-surface condition has been replaced by its linearized approximation. Take the origin of co-ordinates travelling with the surface pressure point, the x-axis horizontal along the track of the disturbance, and the y-axis horizontal at right angles to the x-axis. Write $K = g/U^2$ and define N and θ by the equations

$$Kx = N\cos\theta, \quad Ky = N\sin\theta. \tag{2.1}$$

The critical angle θ_c is defined by

$$\theta_c = \sin^{-1}\tfrac{1}{3} = \tan^{-1}\frac{1}{2\sqrt{2}} \doteqdot 19\tfrac{1}{2}^\circ;$$

and the total force acting on the fluid is denoted by P_0.
Then Peters's expressions for $\zeta(x,y)$ are:

for $x > 0$

$$\zeta(x,y) = \zeta_1(x,y) + \zeta_2(x,y),$$

where

$$\frac{\pi^2 \rho U^2}{P_0}\zeta_1(x,y) = \int_0^\infty dk\, k^2 e^{-kx}\int_0^{\frac{1}{2}\pi} d\gamma \frac{\sin^2\gamma\cos(ky\cos\gamma)}{K^2 + k^2\sin^2\gamma},$$

and

$$-\frac{\pi\rho g}{P_0 K^2}\zeta_2(x,y)$$

$$= 2\lim_{z\to 0}\int_0^\infty (1+u^2)\cos\{Kyu\sqrt{(1+u^2)}\}\sin\{Kx\sqrt{(1+u^2)}\}\exp\{-Kz(1+u^2)\}\,du$$

$$= \operatorname{im}\lim_{z\to 0}\int_{-\infty}^\infty (1+u^2)\exp[iN\{(\cos\theta - u\sin\theta)\sqrt{(1+u^2)}\}]\exp\{-Kz(1+u^2)\}\,du; \tag{2.2}$$

while for $x < 0$, $\zeta(x,y) = \zeta_1(-x,y)$. At the pressure point $x = 0$, $y = 0$ the integrals diverge.

It appears impossible to express ζ_1 and ζ_2 in terms of known functions, but asymptotic expansions for the amplitude at a large distance ($N \gg 1$) behind the pressure point can be found, and these show that the contribution of ζ_1 is negligible inside the critical angle $|\theta| \leqslant \theta_c$. For according to P, p. 142,*

$$\frac{\pi^2\rho K^2 U^2}{P_0}\zeta_1(x,y) = \frac{\pi}{(x^2+y^2)^{\frac{1}{2}}} - \frac{3\pi y^2}{2(x^2+y^2)^{\frac{3}{2}}} - \int_0^\infty dk\, k^4 e^{-kx}\int_0^{\frac{1}{2}\pi} d\gamma\frac{\cos(ky\cos\gamma)}{k^2+K^2\sin^2\gamma}; \tag{2.3}$$

and when the substitution $k = Ku\sin\gamma$ is made in the double integral, this becomes

$$\int_0^\infty du\int_0^{\frac{1}{2}\pi} d\gamma\frac{(K\sin\gamma)^3 u^4}{1+u^2}\exp(-Kxu\sin\gamma)\cos(Kyu\sin\gamma\cos\gamma)$$

$$= \frac{1}{2}\int_0^\pi d\gamma (K\sin\gamma)^3\int_0^\infty du\frac{u^4}{1+u^2}\exp(-Kxu\sin\gamma - iKyu\sin\gamma\cos\gamma)$$

$$= \frac{1}{2}\int_0^\pi d\gamma (K\sin\gamma)^3 F_0(Kx\sin\gamma + iKy\sin\gamma\cos\gamma), \tag{2.4}$$

where by definition

$$F_0(Z) \equiv \int_0^\infty \frac{u^4 e^{-uZ}}{1+u^2}du = \frac{2}{Z^3} - \frac{1}{Z} + \int_0^\infty \frac{e^{-uZ}}{1+u^2}du.$$

* References preceded by the letters H or P are to Hogner (1923) or Peters (1949), respectively.

The last integral is related to the exponential integral (Jeffreys & Jeffreys 1946, p. 443), and it can be shown that for re $Z > 0$ and $|Z| \leqslant 1$ we have

$$|F_0(Z)| < A\,|Z|^{-3};$$

while for re $Z > 0$ and $|Z| > 1$ we have $|F_0(Z)| < A\,|Z|^{-5}$, by the method of steepest descents. To find a bound for (2.4), write it as

$$\frac{1}{2}\int_0^{\frac{1}{6}\pi} + \frac{1}{2}\int_{\frac{1}{6}\pi}^{\frac{5}{6}\pi} + \frac{1}{2}\int_{\frac{5}{6}\pi}^{\pi} = I_1 + I_2 + I_3, \text{ say.}$$

In I_1 introduce the new variable $\beta = Kx\sin\gamma + iKy\sin\gamma\cos\gamma$; then

$$\left|\frac{d\beta}{d\gamma}\right| = |Kx\cos\gamma + iKy\cos 2\gamma| \geqslant \tfrac{1}{2}\,|Kx+iKy| = \tfrac{1}{2}N,$$

and $$|K\sin\gamma| = |\beta(x+iy\cos\gamma)^{-1}| < 2\beta\,|x+iy|^{-1} = 2\beta KN^{-1}.$$

Thus, $$|I_1| \leqslant \frac{1}{2}\int_0^{\infty} |2\beta|^3 K^3 N^{-3} |F_0(\beta)|\, 2N^{-1} |d\beta| < AK^3N^{-4}. \tag{2.5}$$

Similarly, $$|I_3| < AK^3N^{-4}.$$

To find a bound for I_2, deform the path of integration in the γ-plane to pass below $\gamma = \frac{1}{2}\pi$; on the new path, the real part of $Kx\sin\gamma + iKy\sin\gamma\cos\gamma$ is positive, and

$$|Kx\sin\gamma + iKy\sin\gamma\cos\gamma| > A\,|Kx+iKy|$$

which is uniformly large when N is large. Thus

$$|I_2| \leqslant A\int_{\frac{1}{6}\pi}^{\frac{5}{6}\pi} K^3N^{-5}\,|d\gamma| < AK^3N^{-5}, \tag{2.6}$$

and equations (2.5) and (2.6) show that the double integral in (2.3) is $O(K^3N^{-4})$. Thus ζ_1 is of order N^{-3} everywhere, while ζ_2 is of order $N^{-\frac{1}{2}}$ inside the critical angle, and exponentially small outside the critical angle, see P, equations (5.2), (5.4). Thus, ζ_2 is dominant inside the critical angle, ζ_1 outside. We are considering the asymptotic form of $\zeta(x, y)$ for large $N = K|x+iy|$; the expressions (2.3) and P (5.2), (5.4) are adequate except near $\theta = 0$ and $\theta = \pm\theta_c$. (They will not be quoted here; see, however, § 4 below.) We shall now obtain new expansions valid near $\theta = 0$ and $\theta = \theta_c$.

3. The track of the pressure point

We shall have to evaluate (2.2) for small θ, where, by reason of the symmetry of the pattern, we shall take $\theta \geqslant 0$. If we attempt to find the limit in (2.2) by putting $z = 0$ in (2.2) we see that the resulting integral does not converge. This difficulty is avoided if the path of integration is deformed before z is placed equal to 0. The new path is chosen so that

$$\exp\left[iN\{(\cos\theta - u\sin\theta)\sqrt{(1+u^2)}\}\right] \tag{3.1}$$

tends to zero rapidly as $|u| \to \infty$ along the new path, and there are two cases, $\theta = 0$ and $\theta > 0$, which must be considered separately.

(i) If $\theta = 0$, take a path of integration along the straight line from $u = -\infty\exp(\frac{1}{8}\pi i)$ through $u = 0$ to $\infty\exp(\frac{1}{8}\pi i)$, say. Along this path

$\exp\{iN\sqrt{(1+u^2)}\} \to 0$ as $|u| \to \infty$, and $\exp\{-Kz(1+u^2)\}$ is uniformly bounded for all u on the path and all $z \geqslant 0$. Thus, this integral is uniformly convergent as $z \to 0$, and

$$-\frac{\pi\rho g}{P_0K^2}\zeta_2(x,0) = \text{im}\int_{-\infty\exp(\frac{1}{8}\pi i)}^{\infty\exp(\frac{1}{8}\pi i)}(1+u^2)\exp\{iN\sqrt{(1+u^2)}\}\,du. \tag{3.2}$$

(Clearly a good deal of latitude is allowed in the choice of the path.) This may be written in terms of Bessel functions, (P, p. 140, cf. Hogner 1924, p. 7):

$$\zeta_2(x,0) = \frac{P_0K^2}{4\rho g}\{3Y_1(Kx)-Y_3(Kx)\} \sim -\frac{P_0K^2}{\rho g}\left(\frac{2}{\pi Kx}\right)^{\frac{1}{2}}\cos(Kx-\tfrac{1}{4}\pi), \tag{3.3}$$

and so ζ_2 is finite when Kx is positive.

(ii) If $\theta > 0$, take a path of integration along the straight line from $u = -\infty\exp(\frac{1}{8}\pi i)$ to $u = 0$, and thence along the straight line to $u = \infty\exp(-\frac{1}{8}\pi i)$ Along this path the expression (3.1) tends to 0 as $|u| \to \infty$, and $\exp\{-Kz(1+u^2)\}$ is uniformly bounded. Thus, as in (i), the integral is uniformly convergent as $z \to 0$ for fixed θ, and

$$-\frac{\pi\rho g}{P_0K^2}\zeta_2(x,y) = \text{im}\int_{-\infty\exp(\frac{1}{8}\pi i)}^{\infty\exp(-\frac{1}{8}\pi i)}(1+u^2)\exp[iN\{(\cos\theta-u\sin\theta)\sqrt{(1+u^2)}\}]\,du. \tag{3.4}$$

We note that the limits of integration in (3.2) and (3.4) are different, and we are led to suspect singular behaviour near $\theta = 0$. This is confirmed by the following calculation. First, deform the path of (3.4) into two paths of steepest descent through the two saddle points. These points are the roots of

$$(d/du)\{(\cos\theta-u\sin\theta)\sqrt{(1+u^2)}\} = 0,$$

i.e. the roots of $\qquad u\cos\theta-(1+2u^2)\sin\theta = 0,$

i.e. the points

$$u_+(\theta) = \tfrac{1}{4}\{\cot\theta+\sqrt{(\cot^2\theta-8)}\} \quad\text{and}\quad u_-(\theta) = \tfrac{1}{4}\{\cot\theta-\sqrt{(\cot^2\theta-8)}\}, \tag{3.5}$$

which are real if $\theta < \theta_c$ as we may suppose in this section. The paths of steepest descent are sketched in P, fig. 5, in the plane of a convenient variable $\tau = \sinh^{-1}u$ in terms of which the function

$$F(u) \equiv (\cos\theta-u\sin\theta)\sqrt{(1+u^2)} \tag{3.6}$$

is single-valued. The paths of steepest descent $C_-(\theta)$, $C_+(\theta)$ in the cut u-plane are their conformal images and are shown schematically in figure 1.

The path of (3.4) can be deformed into C_-+C_+. Consider the behaviour of the integral as $\theta \to 0$. It is easy to see that the integrand tends to zero uniformly as $|u| \to \infty$ along $C_-(\theta)$, and so, as $\theta \to 0$,

$$\int_{C_-} \to \int_{-\infty\exp(\frac{1}{8}\pi i)}^{\infty\exp(\frac{7}{8}\pi i)}(1+u^2)\exp\{iN\sqrt{(1+u^2)}\}\,du$$

$$= \int_{-\infty\exp(\frac{1}{8}\pi i)}^{\infty\exp(\frac{1}{8}\pi i)}(1+u^2)\exp\{iN\sqrt{(1+u^2)}\}\,du,$$

whence $\qquad \text{im}\int_{C_-} \to -\tfrac{1}{4}\{3Y_1(Kx)-Y_3(Kx)\},$

see equation (3.3)).

As for the integral along $C_+(\theta)$, we note that $u_+(\theta) \sim \frac{1}{2}\cot\theta \to \infty$; thus $C_+(\theta)$ moves off to infinity as $\theta \to 0$. Introduce a new variable $v = u\tan\theta$, in terms of which

$$\int_{C_+} = \cot^3\theta \int_{\infty\exp(\frac{1}{4}\pi i)}^{\infty\exp(-\frac{1}{4}\pi i)} (v^2+\tan^2\theta)\exp\left[\frac{iN\cos^2\theta}{\sin\theta}\{(1-v)\surd(v^2+\tan^2\theta)\}\right]dv. \tag{3.7}$$

The expression $\{\}$ has a saddle point at $v_+(\theta) = u_+(\theta)\tan\theta = \frac{1}{4}(1+\surd(1-8\tan^2\theta))$, which remains finite as $\theta \to 0$; and the expression $(N\cos^2\theta/\sin\theta) = M$ is a large

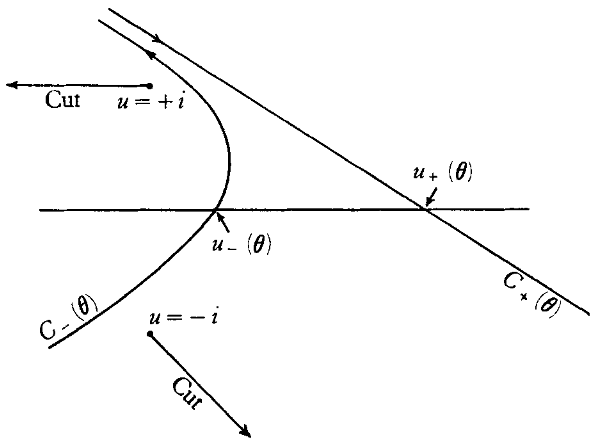

FIGURE 1. Deformation of the contour of integration for small values of θ.

parameter tending to ∞ as $\theta \to 0$ (whether N is large or not). If we apply the ordinary method of steepest descents (see, for example, Jeffreys & Jeffreys 1946, §17.04) to (3.7), we find (the details are omitted) that

$$\int_{C_+} \sim \frac{\cot^3\theta}{4M^{\frac{1}{2}}}\pi^{\frac{1}{2}}\exp\{iMf(\theta)-\tfrac{1}{4}\pi i\}, \quad \text{whence near } \theta = 0$$

$$-\frac{\pi\rho g}{P_0K^2}\zeta_2(x,y) = \operatorname{im}\int_{C_+}$$

$$= \tfrac{1}{4}\pi^{\frac{1}{2}}N^{-\frac{1}{2}}(\sin\theta)^{-\frac{5}{2}}\sin\left\{\frac{N\cos^2\theta}{\sin\theta}f(\theta)-\tfrac{1}{4}\pi\right\}+o(N^{-\frac{1}{2}}(\sin\theta)^{-\frac{5}{2}}), \tag{3.8}$$

where by definition

$$f(\theta) \equiv \frac{3^{\frac{3}{2}}}{16}(1-\tfrac{1}{3}\surd(1-8\tan^2\theta))^{\frac{3}{2}}(1+\surd(1-8\tan^2\theta))^{\frac{1}{2}},$$

$$\to \tfrac{1}{4} \quad \text{as} \quad \theta \to 0.$$

The equation (3.8) shows that the free surface oscillates with infinitely increasing amplitude and infinitely decreasing wavelength as $\theta \to 0$. This behaviour may be compared with the behaviour of waves due to a concentrated initial pressure (Lamb 1932, §239), and is responsible for the infinite wave resistance of the surface pressure point. (For a submerged pressure point or distributed pressure the amplitude on $\theta = 0$ and wave resistance are finite.) We note again that to obtain (3.8) it was not necessary to assume that N is large.

4. The neighbourhood of the critical lines

We have noted in § 1 that existing expansions are inadequate in various ways near $\theta = \pm\theta_c$. On the line $\theta = \theta_c$ itself, the function $F(u)$ of (3.6) has a double zero, and the saddle points $u_+(\theta), u_-(\theta)$ coalesce into a saddle point of higher order at $u = 1/\sqrt{2}$ which is easily treated by the method of steepest descents. Peters (equations (5.3), (5.5) with an error in sign in (5.3)) has shown that

$$-\frac{\pi\rho g}{P_0 K^2}\zeta_2(x, x\tan\theta_c)$$

$$\sim \frac{3^{\frac{4}{3}}}{2^{\frac{3}{2}}}\frac{\Gamma(\frac{1}{3})}{N^{\frac{1}{3}}}\sin\left(\frac{3^{\frac{1}{2}}}{2}N\right) - \frac{3^{\frac{7}{6}}}{2^{\frac{5}{2}}}\frac{\Gamma(\frac{2}{3})}{N^{\frac{2}{3}}}\cos\left(\frac{3^{\frac{1}{2}}}{2}N\right), \tag{4.1}$$

where
$$N = Kx\sec\theta_c.$$

Thus, on (and presumably near) $\theta = \theta_c$, the amplitude decreases as $N^{-\frac{1}{3}}$, while elsewhere it decreases as $N^{-\frac{1}{2}}$ (see P (5.2), (5.4)), and so the waves are prominent near $\theta = \theta_c$ when N is large. We shall now obtain an asymptotic expression which is valid in a finite angle including the line $\theta = \theta_c$. This is to be compared with Hogner's equations (H (82), (83)) which change in form as the line $\theta = \theta_c$ is crossed; it is believed that these hold when $N^{\frac{2}{3}}|\theta - \theta_c|$ is large and $N^{\frac{1}{3}}|\theta - \theta_c|$ is small, but their precise region of validity has not been investigated. Our asymptotic formula will be derived by the method of Chester *et al.* (1957) and will involve the Airy function (see Jeffreys & Jeffreys 1946, § 17.07)

$$\operatorname{Ai}(Z) = \frac{1}{2\pi i}\int_{\infty\exp(-\frac{1}{3}\pi i)}^{\infty\exp(\frac{1}{3}\pi i)} \exp\left(\tfrac{1}{3}t^3 - Zt\right) dt, \tag{4.2}$$

$$= \frac{1}{\pi}\int_0^\infty \cos\left(\tfrac{1}{3}t^3 + Zt\right) dt$$

when Z is real. Since in (4.2) the argument of the exponential is a cubic polynomial, comparison with (3.4) suggests the introduction of a new complex variable v by the implicit relation

$$F(u,\theta) \equiv (\cos\theta - u\sin\theta)\sqrt{(1+u^2)} = -\tfrac{1}{3}v^3 + \mu(\theta)\,v - \nu(\theta), \tag{4.3}$$

where $\mu(\theta)$ and $\nu(\theta)$ are to be chosen so as to make the transformation regular in a uniform neighbourhood of $u = 1/\sqrt{2}$ for all θ sufficiently close to θ_c. The segments of the path of integration of (3.4) lying outside this neighbourhood are deformed into segments of paths of steepest descent from the saddle points, and the contribution from these is negligible, as in the ordinary method of steepest descents. The only significant contribution thus comes from the neighbourhood of the (nearly coincident) saddle points where the transformation (4.3) is applicable.

On differentiating (4.3), keeping θ fixed, we get

$$\frac{\partial F(u,\theta)}{\partial u}\frac{du}{dv} = -v^2 + \mu(\theta). \tag{4.4}$$

The left-hand side vanishes when $u = u_{\pm}(\theta)$, the right-hand side when $v = \pm\mu^{\frac{1}{2}}(\theta)$. If the (u, v)-transformation is to be regular, these points must correspond, and so, from (4.3),

$$\left.\begin{aligned} F(u_+, \theta) &\equiv (\cos\theta - u_+ \sin\theta)\sqrt{(1+u_+^2)} = \tfrac{2}{3}\mu^{\frac{3}{2}}(\theta) - \nu(\theta), \\ F(u_-, \theta) &\equiv (\cos\theta - u_- \sin\theta)\sqrt{(1+u_-^2)} = -\tfrac{2}{3}\mu^{\frac{3}{2}}(\theta) - \nu(\theta), \end{aligned}\right\} \tag{4.5}$$

where $u_+(\theta)$ and $u_-(\theta)$ are known functions of θ, see equation (3·5). These equations are readily solved for $\mu(\theta)$ and $\nu(\theta)$

$$\mu^{\frac{3}{2}}(\theta) = \frac{3^{\frac{5}{2}}}{64}\frac{\cos^2\theta}{\sin\theta}\{(1+Q)^{\frac{1}{2}}(1-\tfrac{1}{3}Q)^{\frac{3}{2}} - (1-Q)^{\frac{1}{2}}(1+\tfrac{1}{3}Q)^{\frac{3}{2}}\}, \tag{4.6}$$

$$\nu(\theta) = -\frac{3^{\frac{3}{2}}}{32}\frac{\cos^2\theta}{\sin\theta}\{(1+Q)^{\frac{1}{2}}(1-\tfrac{1}{3}Q)^{\frac{3}{2}} + (1-Q)^{\frac{1}{2}}(1+\tfrac{1}{3}Q)^{\frac{3}{2}}\}, \tag{4.7}$$

where $Q \equiv \sqrt{(1-8\tan^2\theta)}$. By expanding these expressions it can be shown that $\mu(\theta)$ and $\nu(\theta)$ involve only even powers of Q and are therefore regular functions of θ.

This also follows from the general theory (Chester *et al.* 1957), which shows further that with these values of $\mu(\theta)$ and $\nu(\theta)$ the transformation (4.3) is indeed uniformly regular, and that the points $u = u_{\pm}(\theta)$, $v = \pm\mu^{\frac{1}{2}}(\theta)$ correspond. Following the procedure explained by Chester *et al.* (1957) we change the variable of integration in (3.4) to v, and expand $(1+u^2)\, du/dv$ in the form

$$(1+u^2)\frac{du}{dv} = \sum_0^\infty p_m(\theta)\,(v^2-\mu(\theta))^m + v\sum_0^\infty q_m(\theta)\,(v^2-\mu(\theta))^m,$$

which holds uniformly when v and $\theta - \theta_c$ are sufficiently small. The theory shows that the asymptotic expansion of (3.4) is

$$\Sigma p_m(\theta)\int (v^2-\mu(\theta))^m \exp[iN(-\tfrac{1}{3}v^3+\mu v-\nu)]\,dv$$
$$+\Sigma q_m(\theta)\int v(v^2-\mu(\theta))^m \exp[iN(-\tfrac{1}{3}v^3+\mu v-\nu)]\,dv,$$

where the integration can be extended with a negligible error from $-\infty\exp(\frac{1}{6}\pi i)$ to $\infty\exp(-\frac{1}{6}\pi i)$, cf. P, fig. 6. To obtain the dominant terms it is sufficient to calculate the leading coefficients $p_0(\theta)$ and $q_0(\theta)$. On putting $u = u_{\pm}(\theta)$, $v = \pm\mu^{\frac{1}{2}}(\theta)$, we find that

$$(1+u_{\pm}^2)\left(\frac{du}{dv}\right)_{\pm} = p_0(\theta) \pm \mu^{\frac{1}{2}}(\theta)\,q_0(\theta),$$

and the left-hand side is known when du/dv is known at $v = \pm\mu^{\frac{1}{2}}(\theta)$.

But, from (4.4),

$$\frac{\partial^2 F}{\partial u^2}\left(\frac{du}{dv}\right)^2 + \frac{\partial F}{\partial u}\frac{d^2u}{dv^2} = -2v,$$

whence

$$\left(\frac{\partial^2 F}{\partial u^2}\right)_{\pm}\left(\frac{du}{\partial v}\right)_{\pm}^2 = \mp 2\mu^{\frac{1}{2}}, \tag{4.8}$$

where

$$\left(\frac{\partial^2 F}{\partial u^2}\right)_{\pm} = \mp\frac{4}{3^{\frac{1}{2}}}\sin\theta\, Q(1\pm Q)^{-\frac{1}{2}}(1\mp\tfrac{1}{3}Q)^{-\frac{1}{2}}, \tag{4.9}$$

as is easily shown. The sign of $(du/dv)_{\pm}$ is positive for small $\theta-\theta_c$, since $u = u_{\pm}$, $v = \pm\mu^{\frac{1}{2}}$ correspond. Thus, from (4.8) and (4.9),

$$(1+u_{\pm}^2)\left(\frac{du}{dv}\right)_{\pm} = \frac{3^{\frac{5}{4}}}{2^{\frac{9}{2}}}\frac{\cos^2\theta}{\sin^{\frac{5}{2}}\theta}\left(\frac{\mu(\theta)}{1-8\tan^2\theta}\right)^{\frac{1}{4}}(1\pm Q)^{\frac{5}{4}}(1\mp\tfrac{1}{3}Q)^{\frac{5}{4}},$$

whence

$$p_0(\theta) = \frac{3^{\frac{5}{4}}}{2^{\frac{11}{2}}}\frac{\cos^2\theta}{\sin^{\frac{5}{2}}\theta}\left(\frac{\mu(\theta)}{1-8\tan^2\theta}\right)^{\frac{1}{4}}\{(1+Q)^{\frac{5}{4}}(1-\tfrac{1}{3}Q)^{\frac{5}{4}}+(1-Q)^{\frac{5}{4}}(1+\tfrac{1}{3}Q)^{\frac{5}{4}}\} \tag{4.10}$$

and

$$q_0(\theta) = \frac{3^{\frac{5}{4}}}{2^{\frac{11}{2}}}\frac{\cos^2\theta}{\sin^{\frac{5}{2}}\theta}\left(\frac{\mu(\theta)}{1-8\tan^2\theta}\right)^{-\frac{1}{4}}Q^{-1}\{(1+Q)^{\frac{5}{4}}(1-\tfrac{1}{3}Q)^{\frac{5}{4}}-(1-Q)^{\frac{5}{4}}(1+\tfrac{1}{3}Q)^{\frac{5}{4}}\}, \tag{4.11}$$

where $Q = N(1-8\tan^2\theta)$; the functions $p_0(\theta)$ and $q_0(\theta)$ are regular near $\theta = \theta_c$. We thus see that the integral (3.4) is asymptotically

$$p_0(\theta)\int\exp\left[iN(-\tfrac{1}{3}v^3+\mu v-\nu)\right]dv + q_0(\theta)\int v\exp\left[iN(-\tfrac{1}{3}v^3+\mu v-\nu)\right]dv$$

$$= -ip_0(\theta)\int\exp\left[N(\tfrac{1}{3}w^3+\mu w-i\nu)\right]dw - q_0(\theta)\int w\exp\left[N(\tfrac{1}{3}w^3+\mu w-i\nu)\right]dw,$$

where $w = iv$ and the integration is from $w = \infty\exp(-\frac{1}{3}\pi i)$ to $w = \infty\exp(\frac{1}{3}\pi i)$. But these integrals are multiples of the Airy function (4.2) and of its derivative. Thus the integral in (3.4) is asymptotically

$$2\pi i\exp(-iN\nu(\theta))\{-iN^{-\frac{1}{3}}p_0(\theta)\,\mathrm{Ai}(-N^{\frac{2}{3}}\mu(\theta)) + N^{-\frac{2}{3}}q_0(\theta)\,\mathrm{Ai}'(-N^{\frac{2}{3}}\mu(\theta))\}.$$

Higher terms in the expression are of order $N^{-\frac{4}{3}}\,\mathrm{Ai}$, $N^{-\frac{5}{3}}\,\mathrm{Ai}'$ at most. The amplitude $\zeta_2(x,y)$ is obtained from the imaginary part, cf. equation (3.4)

$$\frac{\rho g}{2P_0K^2}\zeta_2(x,y) \sim \frac{p_0(\theta)}{N^{\frac{1}{3}}}\mathrm{Ai}(-N^{\frac{2}{3}}\mu(\theta))\sin(N\nu(\theta)) - \frac{q_0(\theta)}{N^{\frac{2}{3}}}\mathrm{Ai}'(-N^{\frac{2}{3}}\mu(\theta))\cos(N\nu(\theta)), \tag{4.12}$$

where $\mu(\theta)$, $\nu(\theta)$, $p_0(\theta)$, $q_0(\theta)$ are defined by equations (4.6), (4.7), (4.10), (4.11), respectively, and are regular near $\theta = \theta_c$. It can be shown that near $\theta = \theta_c$.

$$\left.\begin{aligned}\mu(\theta) &= -\frac{3}{2^{\frac{1}{2}}}(\theta-\theta_c)+O((\theta-\theta_c)^2),\\ \nu(\theta) &= -\frac{3^{\frac{1}{2}}}{2}+\frac{3^{\frac{1}{2}}}{2^{\frac{1}{2}}}(\theta-\theta_c)+O((\theta-\theta_c)^2),\end{aligned}\right\} \tag{4.13}$$

and that

$$p_0(\theta_c) = \frac{3^{\frac{3}{2}}}{2^{\frac{3}{2}}}, \quad q_0(\theta_c) = \frac{15}{2^{\frac{5}{2}}}.$$

Values of the functions $\mu(\theta), \nu(\theta), p_0(\theta), q_0(\theta)$ are shown in Table 1. The functions Ai and Ai′ are tabulated by Miller (1946).

Equation (4.12) gives the wave pattern far behind the travelling pressure point; from the general theory (Chester *et al.* 1957) it follows that it is valid in

θ (degrees)	$\mu(\theta)$	$\nu(\theta)$	$p_0(\theta)$	$q_0(\theta)$
10·00	0·532	−1·227	9·484	11·131
10·25	0·510	−1·210	8·890	10·545
10·50	0·488	−1·193	8·348	10·002
10·75	0·468	−1·177	7·852	9·500
11·00	0·448	−1·162	7·397	9·035
11·25	0·429	−1·147	6·979	8·603
11·50	0·410	−1·133	6·594	8·201
11·75	0·392	−1·120	6·239	7·826
12·00	0·374	−1·107	5·911	7·476
12·25	0·357	−1·095	5·670	7·148
12·50	0·340	−1·083	5·326	6·842
12·75	0·324	−1·071	5·065	6·555
13·00	0·308	−1·060	4·822	6·285
13·25	0·293	−1·050	4·596	6·032
13·50	0·278	−1·039	4·385	5·793
13·75	0·263	−1·030	4·189	5·569
14·00	0·249	−1·020	4·005	5·357
14·25	0·235	−1·011	3·833	5·157
14·50	0·221	−1·002	3·672	4·967
14·75	0·208	−0·993	3·521	4·788
15·00	0·195	−0·984	3·379	4·619
15·25	0·182	−0·976	3·246	4·458
15·50	0·169	−0·968	3·120	4·306
15·75	0·157	−0·961	3·002	4·161
16·00	0·145	−0·953	2·891	4·023
16·25	0·133	−0·946	2·786	3·892
16·50	0·122	−0·939	2·686	3·768
16·75	0·111	−0·932	2·593	3·649
17·00	0·100	−0·925	2·504	3·536
17·25	0·089	−0·918	2·420	3·428
17·50	0·078	−0·912	2·340	3·324
17·75	0·067	−0·906	2·265	3·226
18·00	0·057	−0·899	2·193	3·131
18·25	0·047	−0·894	2·125	3·041
18·50	0·037	−0·888	2·060	2·955
18·75	0·027	−0·882	1·999	2·872
19·00	0·018	−0·876	1·940	2·793
19·25	0·008	−0·871	1·884	2·716
19·50	−0·001	−0·865	1·831	2·643
19·75	−0·010	−0·860	1·780	2·573
20·00	−0·019	−0·855	1·732	2·506

TABLE 1. The coefficient functions $\mu(\theta)$, $\nu(\theta)$, $p_0(\theta)$, $q_0(\theta)$ occurring in equation (4.12) for the elevation

some finite angle including $\theta = \theta_c$ and it reduces to the equations of Peters (P (5.2), (5.4)) when $|N^{\frac{2}{3}}\mu(\theta)|$ is large, as can be shown by using the asymptotic expressions

$$\left.\begin{aligned} \mathrm{Ai}(X) &\sim \tfrac{1}{2}\pi^{-\frac{1}{2}}X^{-\frac{1}{4}}\exp\left(-\tfrac{2}{3}X^{\frac{3}{2}}\right), \quad \mathrm{Ai}'(X) \sim -\tfrac{1}{2}\pi^{-\frac{1}{2}}X^{\frac{1}{4}}\exp\left(-\tfrac{2}{3}X^{\frac{3}{2}}\right), \\ \mathrm{Ai}(-X) &\sim \pi^{-\frac{1}{2}}X^{-\frac{1}{4}}\cos\left(\tfrac{2}{3}X^{\frac{3}{2}}-\tfrac{1}{4}\pi\right), \quad \mathrm{Ai}'(-X) \sim \pi^{-\frac{1}{2}}X^{\frac{1}{4}}\cos\left(\tfrac{2}{3}X^{\frac{3}{2}}-\tfrac{3}{4}\pi\right), \end{aligned}\right\} \quad (4.14)$$

valid when X is large and positive. The equation (P (5.5)) is obtained by supposing that $N^{\frac{2}{3}}\mu(\theta)$ is small, expanding the Airy function in powers of $\theta-\theta_c$ and retaining only the first two terms.

5. Description of the wave pattern

Let us consider first the well-known pattern some distance inside the critical lines where the Airy functions may be replaced by their asymptotic expansions in terms of circular functions. When N is large, the surface elevation in this region is of the form (see P (5.2))

$$\frac{A_1(\theta)}{N^{\frac{1}{2}}}\cos\{Nf_1(\theta)+\epsilon_1\}+\frac{A_2(\theta)}{N^{\frac{1}{2}}}\cos\{Nf_2(\theta)+\epsilon_2\}. \tag{5.1}$$

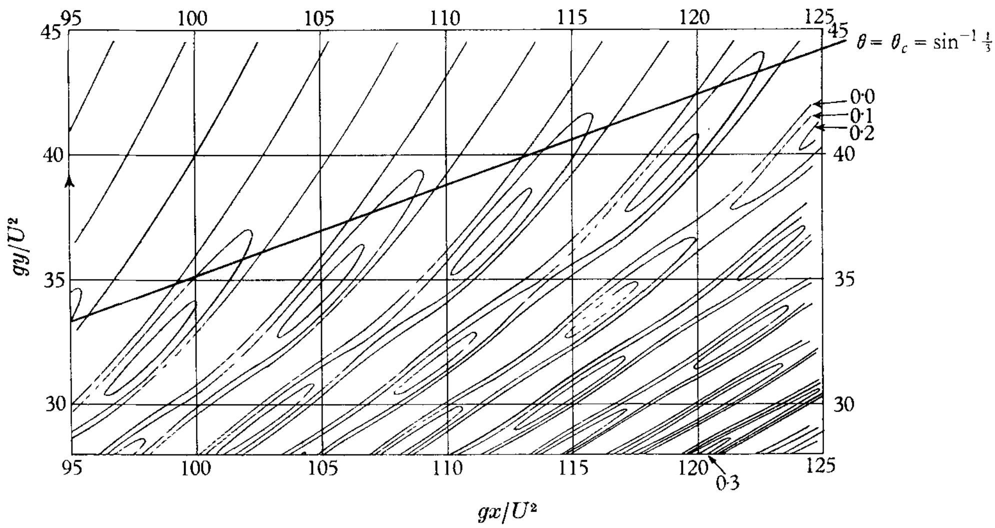

FIGURE 2. Contours of equal magnitude of the dimensionless surface elevation $\rho U^4\zeta_2(x,y)/2gP_0$ due to a concentrated force P_0 moving with velocity U, for $gr/U^2 = 2\pi r/\lambda$ near 100. (Here r denotes the distance from the disturbance, λ the wavelength of waves moving with phase velocity U.) For the sake of clarity only positive values of the elevation are shown. Note that the largest elevation in the figure occurs near the point (120·2, 28), not near $\theta = \theta_c$.

The curves of constant phase corresponding to each term are of the form

$$rf(\theta) = \text{const.};$$

that is, they are geometrically similar with respect to the origin (see also, Lamb 1932, §256). The amplitude of each term falls off like $r^{-\frac{1}{2}}$. The total surface elevation, being the sum of two terms, does not exhibit exact similarity with respect to the origin, but the amplitude is almost periodic along radii from the origin.

Next let us consider the surface elevation (4.12) near the critical line $\theta = \theta_c$ when N is large. For very large N and bounded values of $N^{\frac{2}{3}}|\theta-\theta_c|$ the first term in (4.12) is dominant. Thus, the nodal lines (contours of zero surface elevation) lie near those curves where the first term vanishes, that is, near those curves where either $\sin(N\nu(\theta))$ or $\mathrm{Ai}(-N^{\frac{2}{3}}\mu(\theta))$ vanishes, that is,

$$\text{near } N\nu(\theta) = m\pi, \quad \text{where } m \text{ is any large integer,} \tag{5.2}$$

and

$$\text{near } N^{\frac{2}{3}}\mu(\theta) = |a_1|, |a_2|, \ldots, |a_s|, \ldots, \tag{5.3}$$

where $a_s < 0$ is the sth zero of Ai (z). It follows from (4.13) that the curves (5.2) are ultimately equidistant straight lines crossing $\theta = \theta_c$ at an angle $\tan^{-1} 1/\sqrt{2}$ that is, inclined at an angle $\tan^{-1}\sqrt{2} = \frac{1}{4}\pi + \frac{1}{2}\theta_c \doteqdot 54\frac{3}{4}°$, to the track of the disturbance, as was stated by Kelvin (W. Thomson 1891, p. 485). As for the curves (5.3), it follows from (4.13) that they are approximately of the form

$$Kp \doteqdot \frac{2^{\frac{1}{2}}}{3}(Kr)^{\frac{1}{3}}\,|a_s|,$$

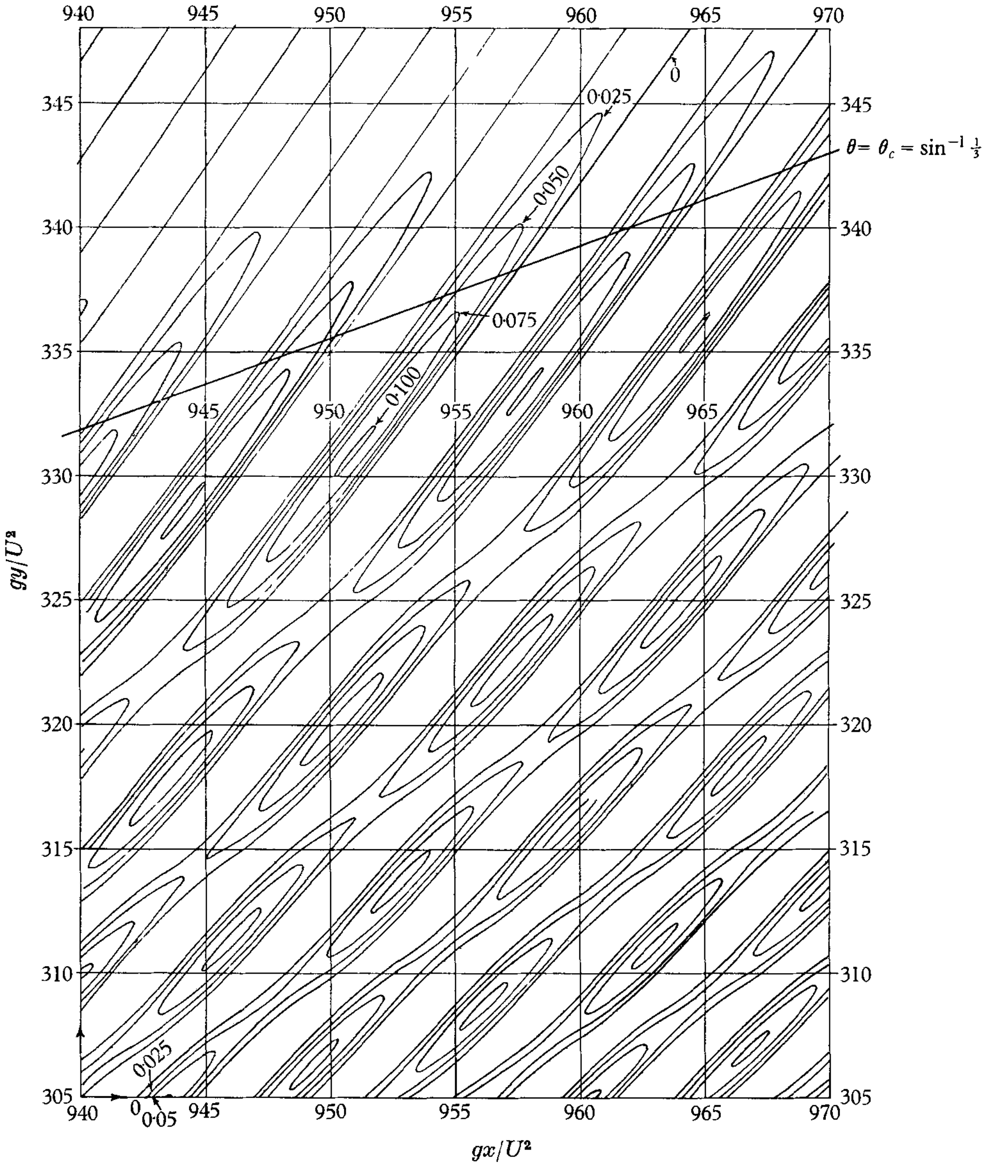

FIGURE 3. Contours of equal magnitude of the dimensionless surface elevation $\rho U^4\zeta_2(x, y)/2gP_0$, for gr/U^2 near 1000. For the sake of clarity only positive values of the elevation are shown. The elevation again exceeds 0·08 near the point (963, 285).

where p is the perpendicular distance from $\theta = \theta_c$ and r is the distance from the origin. These curves lie inside the critical lines, and it is seen that p increases only slowly with distance from the origin. Thus, the curves (5.2) and (5.3) combine to form a net of (approximate) parallelograms: the sides parallel to $\theta = \theta_c$ are of nearly constant length, while the length of the sides parallel to $\theta = \frac{1}{4}\pi + \frac{1}{2}\theta_c$ increases as the cube root of the distance. The amplitude due to the first term in (4.12) varies with distance as $N^{-\frac{1}{3}}$ and this is ultimately much larger (by a factor

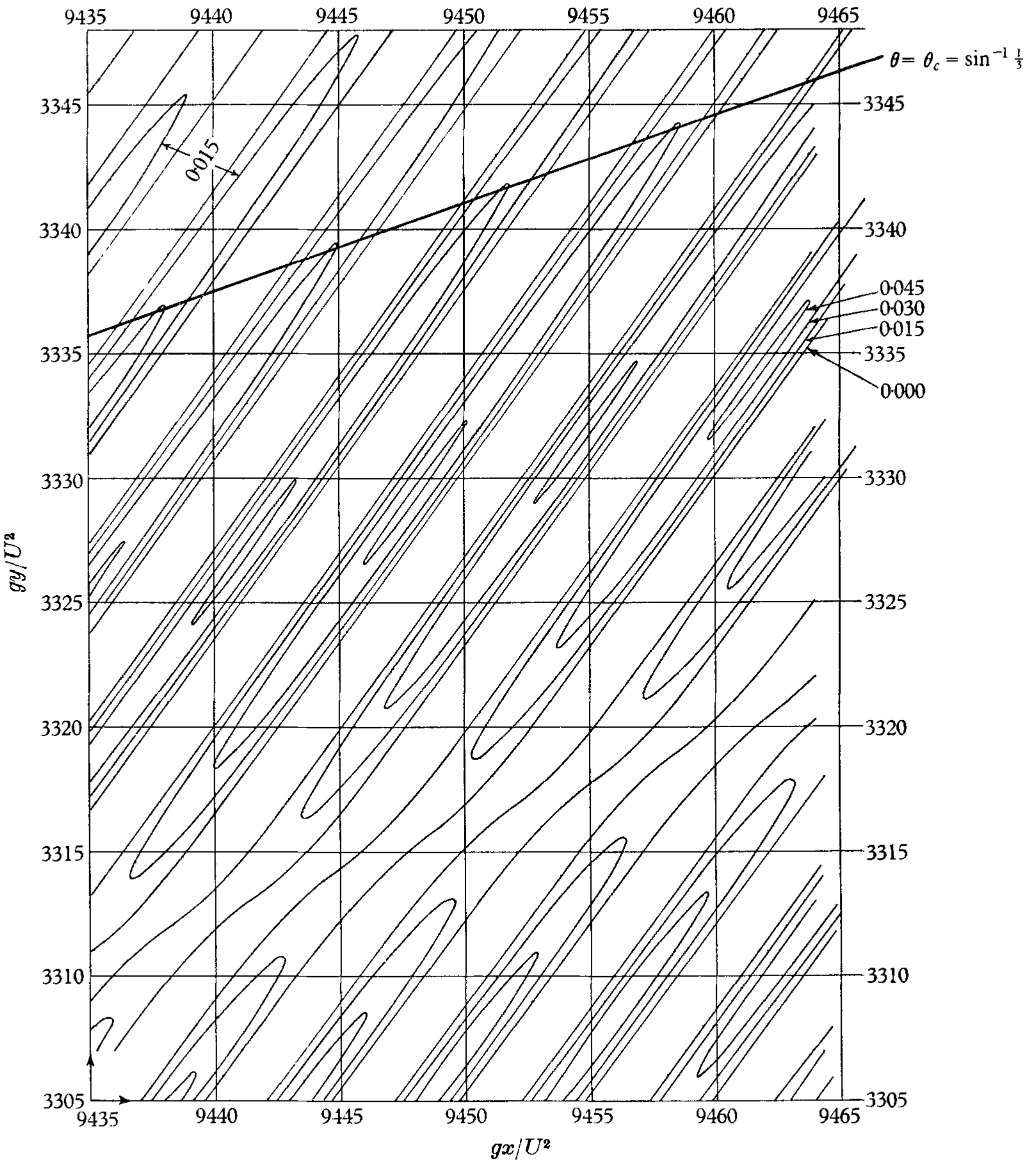

FIGURE 4. Contours of equal magnitude of the dimensionless surface elevation $\rho U^4\zeta_2(x, y)/2gP_0$, for gr/U^2 near 10,000. For the sake of clarity only positive values of the elevation are shown. Note the progressive lengthening of the crests with distance from the disturbance, as shown by figures 2 to 4.

$N^{\frac{1}{3}}$) than the term $N^{-\frac{1}{2}}$ describing the variation with distance well inside the critical angle. Thus, ultimately the largest waves are near the critical lines $\theta = \pm\theta_c$ and near $\theta = 0$ where the linearized theory predicts infinite amplitude and infinitesimal wave length.

Figures 2, 3 and 4 show contours of equal surface elevation computed from (4.12) for three distances, near $N = 100$, 1000 and 10,000, respectively, where $N = gr/U^2 = 2\pi r/\lambda$, and λ is the wavelength of a regular wave-train travelling with phase velocity U. In each case the nodal lines lie near the boundaries of the parallelograms mentioned above. There are striking phase changes near the zeros of Ai $(-N^{\frac{2}{3}}\mu(\theta))$; these arise from the presence of the second term in (4.12). The lengthening of the crests with increasing N is clearly shown. Figure 3 ($N \doteqdot 1000$) resembles closely the contour map computed by Hogner (1923).

In figure 2 ($N \doteqdot 100$) the largest amplitudes occur near the point (120·2, 28) which is not near $\theta = \theta_c$ where they would be expected. The reason is that the quotient $N^{-\frac{1}{6}}$ obtained in the last paragraph but one is about 0·46 and differs little from unity. The reduction in wave height by this factor is more than compensated by the fairly rapid increase in $p_0(\theta)$ and $q_0(\theta)$ as θ decreases from θ_c; see Table 1 above. Even when N is near 10,000, the quotient $N^{-\frac{1}{6}}$ is still about 0·22. Thus the concentration of amplitude near the critical lines is not very striking (much less so than is suggested by Havelock's comparison (1908, Table 2) of the amplitude on the critical lines with the amplitude of transverse waves on $\theta = 0$).

In the computations it is assumed that the pressure is concentrated at a point on the free surface. If the pressure is distributed over an area, the resulting amplitude is obtained by an integration over the area, and in regions where the wavelength is short (compared to the dimensions of the pressure area) the amplitude will be reduced by destructive interference. In particular the infinite amplitude predicted near the track of the disturbance will not be observed. Similar remarks apply when the waves are due to a submerged instead of a surface source.

Mr H. P. F. Swinnerton-Dyer computed for me the co-ordinates of points on the contours of equal elevation which are shown in figures 2, 3 and 4. His help and advice is most gratefully acknowledged.

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